General $\mathcal{K} = -1$ Friedman-Lemaître models and the averaging problem in cosmology.

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We introduce the notion of general $\mathcal{K} = -1$ Friedman-Lemaître (compact) cosmologies and the notion of averaged evolution by means of an averaging map. We then analyze the Friedman-Lemaître equations and the role of gravitational energy on the universe evolution. We distinguish two asymptotic behaviors: radiative and mass gap. We discuss the averaging problem in cosmology for them through precise definitions. We then describe in quantitative detail the radiative case, stressing on precise estimations on the evolution of the gravitational energy and its effect in the universe's deceleration. Also in the radiative case we present a smoothing property which tells that the long time $H^3 \times H^2$ stability of the flat $\mathcal{K} = -1$ FL models implies $H^{i+1} \times H^i$ stability independently of how big the initial state was in $H^{i+1} \times H^i$, i.e. there is long time smoothing of the space time[a]. Finally we discuss the existence of initial "big-bang" states of large gravitational energy, showing that there is no mathematical restriction to assume it to be low at the beginning of time.

[a]The word smoothing here is referred to the decay toward zero of the space time Bel-Robinson curvatures (and therefore of the derivatives), and not to a gain in Sobolev regularity as in usual PDE terminology.

1 Introduction

An implicit assumption of the Friedman-Lemaître cosmologies as models of the actual universe is that, because matter distribution at large scales (visible or not) appears to be "to a good extent" homogeneous and isotropic, the large scale evolution of the universe should be modeled as driven "to a good extent" by an exactly homogeneous and isotropic material distribution. The assumption, now known as the *averaging problem in cosmology*, needs quantitative approval or disproval (see [1]). Phrasing the problem in a question one asks: is the large scale evolution affected by the small scale structure?. The reason of the difficulty lies evidently in the nonlinearity of the Einstein equation. An averaged source of matter doesn't give rise necessarily to the average of the original solution. We will discuss this and other issues from the perspective of *general cosmological models*, i.e. the study of arbitrary solutions of the Einstein equation in the Hubble gauge (constant mean curvature (CMC)

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gauge) provided with a set of Friedman-Lemaître equations giving the cosmological interpretation to the framework.

The standard $\mathcal{K} = -1$ FL cosmology describes the universe history by the evolution of the energy and pressure densities of the different type of matter present. Starting from a "big-bang" where the densities and the space-time curvature blow up, the universe evolution is described as eternally expanding, with decaying densities and space-time curvature at a particular pace according to their matter type. Such a description is analytically possible due to the homogeneity and isotropy of the space which reduce the Einstein equations into a set of ordinary differential equations, the so called Friedman-Lemaître equations. We will deal here with compact cosmologies, i.e. space-times with compact Cauchy surfaces of hyperbolic type. When speaking about homogeneity and isotropy of a compact cosmology we will refer to those properties in the universal cover solution. In its formal terms the geometric structure of the space-time is described by a metric of the form $\mathbf{g} = -d\tau^2 + a^2(\tau)/V_H^{\frac{2}{3}} g_H$ on a four-manifold $\mathbb{R} \times \Sigma$ where Σ is a compact hyperbolic manifold i.e. a manifold admitting a metric of constant negative sectional curvature and where V_H is the volume of Σ endowed with the unique hyperbolic metric (the one with sectional curvature equal to negative one). If the densities of energy and pressure of the material fields are $\rho(\tau)$ and $p(\tau)$ the FL equations are

$$\mathcal{H}^2 = \frac{8\pi G\rho}{3} - \frac{\mathcal{K}V_H^{\frac{2}{3}}}{a^2}, \tag{1}$$

$$\frac{a''}{a} = -\frac{4\pi G(\rho + 3p)}{3}, \tag{2}$$

where $\mathcal{H} = a'/a$ is the Hubble parameter and G is the gravitational constant. These equations must be complemented with an equation of state $p(\rho)$. An obvious observation about these models is that they do not have any pure gravitational degree of freedom besides the gravitational field generated by the matter present. This fact is seen by making $\rho = p = 0$ and observing that in that case the solutions are flat. We call these flat solutions *flat cones* as they can be obtained as quotients of a future light cone in Minkowski space-time. For non homogeneous and isotropic solutions there is no way to define which part of the gravitational field is generated and which part is free, as those properties (if anything) would be potentially defined only in special solutions or in asymptotic regimes. There are simply two fields interacting, gravitation and matter. In this sense, the gravitational field adds a new degree of freedom to general cosmological models which needs to be quantitatively described.

We have found that a satisfactory way to analyze arbitrary solutions to the Einstein equations on the light of the questions raised by cosmology and those raised by the FL models themselves, is to introduce the notion of *general cosmological model: an arbitrary solution to the Einstein equations in the Hubble gauge, provided with a set of Friedman-Lemaître equations giving its interpretative cosmological meaning.* Unlike the FL models where the FL equations are enough to describe the evolution, in general cosmological models one must rely on the full Einstein equations to predict

the behavior of the terms involved in the general FL equations and therefore interpret the solutions in cosmological terms. One purpose of the article is to start a rigorous analysis of the general FL equations using the full Einstein equations.

An arbitrary solution $\mathbf{g}$ on a space-time manifold $\mathbb{R}\times\Sigma$ where Σ is a compact hyperbolic manifold, is in the Hubble gauge if the mean extrinsic curvature k of the equal time Cauchy surfaces is constant. The foliation $\mathbb{R}\times\Sigma$ is called the CMC foliation. It is well know to be unique, intrinsically defined, an with the mean curvature k varying monotonically on it, in particular k or $\mathcal{H}$ (as we will se the Hubble parameter is $\mathcal{H}=\frac{-k}{3}$) can be taken as a time variable. It is important to remark that unlike other gauges, the Hubble gauge is intrinsic, i.e. it is implicitly given by the solution. Let's write the metric as

$$\mathbf{g}=-(N)^2dk^2+X^*\times dk+dk\otimes X^*+g, \tag{3}$$

where N is the *lapse function*, X the *shift vector* and g is a spatial three-metric on Σ. To write the general FL equations one defines the radius $a(k)$ at the time k as $a(k)=V(k)^{\frac{1}{3}}$ and the proper time $\tau(k)$ at the time k through (see [2] for a related approach)

$$\frac{d\tau}{dk}=\frac{\int Ndv_g}{V}. \tag{4}$$

With these definitions the FL equations (deduced from the Einstein equations, see subsection 3.1) are

1. First FL equation : $\mathcal{H}^2=-\dfrac{\int_\Sigma \mathcal{N}Rdv_g}{6V}+\dfrac{\int_\Sigma \mathcal{N}(16\pi G\rho+|\hat{K}|^2)dv_g}{6V}$, (5)

2. Second FL equation : $\dfrac{a''}{a}=\dfrac{-\int_\Sigma \mathcal{N}(4\pi G(\rho+3p)+|\hat{K}|^2)dv_g}{3V}$. (6)

Where $\mathcal{N}=\frac{N}{\bar{N}}$ (bar denotes volume-average) and has average equal to one. The derivatives, denoted with a prime are proper time derivatives, i.e. $'=\frac{d}{d\tau}$. $\hat{K}$ is the traceless part of the second fundamental form K. Compared with the second FL equation 2 in a perfect FL cosmology we observe the appearance of the weight term $\mathcal{N}$ which inexorably couples matter to gravitation and a purely gravitational term of $|\hat{K}|^2$ which is essential and represents the additional gravitational degree of freedom mentioned before. A particular solution is a FL model iff $\hat{K}=0$ and $\mathcal{N}=1$.

On the light of general cosmological models, a fundamental question is to quantify the evolution of the different terms that appear in the FL equations. It is important to realize that the ultimate goal would be to understand the FL equations for solutions which are realistic at small scales, i.e. at the natural scale of the flow. This is a difficult problem, however we will argue that we can have an starting point if precise assumptions are made. Namely, in subsection 3.5 we will introduce *assumption* (C), a precise quantitative hypothesis on the behavior of arbitrary solutions at late times, from which we will make explicit estimations of the different terms involved in the FL equations. Assumption (C) is a close relative of the *weak cosmic censorship conjecture* of Penrose, conjecture stated in an

asymptotically flat context. In rough terms, assumption (C) precisely describes a family of solutions and divides it in two classes: *radiation* and *mass gap*. A radiative solution is an ideal solution in which no sort of compact object emerges along evolution, i.e. universes filled only with radiation. We will study this case in detail, although only for gravitational radiation. The technique may be applied to other radiative contexts as well. In this case the gravitational field can safely be isolated from the rest, and one can safely interpret $\frac{|\hat{K}|^2}{16\pi G} = \rho_G = p_G$ as the effective energy and pressure densities of gravitational radiation. These densities are quantitatively studied along with the decay of $\mathcal{N}$ to one. The estimates are given in Theorem 1 (see statement below) which in addition give estimates on the Bel-Robinson energies Q_i. Altogether Theorem 1 provides a detailed structure of the radiative solutions and is one of the main result of this article.

Theorem 1 (Expansive smoothing and energy estimates). *Let Σ be a compact and rigid hyperbolic manifold. There is an $\epsilon > 0$ such that the Einstein CMC flow of a cosmologically scaled initial state (i.e. with $\mathcal{H} = 1$) (g, K) with $\mathcal{V} - \mathcal{V}_{inf} \leq \epsilon$ and $\tilde{\mathcal{E}}_1 \leq \epsilon$ has the following long time properties (take $t = \frac{1}{\mathcal{H}}$):*

1. *The limit $lim_{t\rightarrow\infty} t^3 Q_0$ is finite and greater than zero.*
2. *there are $n_i \geq 0$ such that $\lim_{t\rightarrow\infty} \frac{t^{2i+3}}{(\ln t)^{n_i}} Q_i \leq \infty$ for $i \geq 1$.*
3. *for given $\gamma > 0$, $\int_t^\infty \frac{\int_\Sigma |\hat{K}|^2 dv_g}{u} du \geq C t^{-(2+\gamma)}$.*
4. *$|\hat{K}|^2 \leq C t^{-4}$ pointwise (not volume averaged).*

In particular the cosmologically scaled flow of a $H^i \times H^{i-1}$ state (for any $i \geq 1$) as in the hypothesis above converges in $H^i \times H^{i-1}$ to the canonical flat cone state $(g, K) = (g_H, -g_H)$.

Theorem 1 is in PDE terminology a small data statement. The small data condition is stated as saying that the reduced volume $\mathcal{V} = \mathcal{H}^3 V$ is ϵ-close to its infimum and the first Bel-Robinson energy $\mathcal{E}_1$ ϵ-close to zero. These two conditions can be seen to be equivalent [3] to the statement that the initial data (g, K) is close in the Sobolev space $H^3 \times H^2$ to the flat cone state $(g_H, -g_H)$ where g_H is the unique hyperbolic metric (up to diffeomorphism). A hyperbolic manifold is called rigid if it doesn't admit traceless Codazzy tensors (see [6] for a discussion). The topological condition of rigidity is important to get the precise estimates above. It is possible to get estimates in the non rigid case but they are different, in particular those on the gravitational energy. The importance of rigidity is that it allows the control of the H^2 norm of harmonic metrics with respect to the hyperbolic metric (spatial gauge) only by their Ricci tensor.

The estimates in theorem 1 are compatible with what one would expect is a radiative behavior. According to the standard FL models an exact radiative behavior would imply a pointwise decay on

the gravitational energy density of the form

$$\frac{|\hat{K}|^2}{16\pi G} \approx \frac{1}{t^4} \tag{7}$$

The estimate in items 3 and 4 in theorem 1 says that in some averaged sense the global gravitational energy decays with a rate between the radiative t^{-1} and the faster t^{-2}. It would be interesting to improve (if possible) the estimate from below in item 3.

In rough terms the *mass gap* solutions can be described as those for which after a sufficiently long time there appear a finite set of isolated stationary solutions separating from each other and with radiation in between. This qualitative description is made quantitative in *assumption (C)*. We analyze the averaging problem for these *mass gap*-solutions. A convenient setup for the analysis is to define the notion of averaged space, a Lorentzian manifold constructed out the averaged parameters $a(k)$ and $\tau(k)$ of the original solution. The averaging problem can be stated as asking to which extent the averaged space is close to a FL model. A remarkable consequence of applying assumption (C) is that the second FL equation is estimated as

$$\frac{a''}{a} = -\frac{4\pi G(\bar{\mathcal{M}}_{ADM} + \bar{\rho} + \bar{\rho}_G + 3(\bar{p} + \bar{p}_G))}{3} + O(t^{-(3+\epsilon)}) \tag{8}$$

where $\bar{\mathcal{M}}_{ADM}$ is the volume average of the ADM masses of the emerging stationary solutions, and $\bar{\rho}, \bar{\rho}_G, \bar{p}, \bar{p}_G$ are the volume averages of the densities of energy and pressure of material and gravitational radiation respectively, filling the space in between. However one must remark that despite all the satisfactory equation 8 may look, it is based on an idealized assumption and on its apriori estimates which so far need to be justified. Also the quantitative description they provide is only asymptotically in time, and not throughout the full evolution.

A quantity underlying all the averaging formalism is the so called *reduced volume* [9], defined above as $\mathcal{V} = \mathcal{H}^3 V$. It decreases monotonically, and is bounded below by the topological invariant V_H. It has been used in [3] to show the long time geometrization of the Einstein flow under curvature bounds. Here it is manifested throughout the article in different forms. Its monotonicity is shown to be equivalent to the universes deceleration and is used to get the estimate in item 3 in Theorem 1. We will introduce an use an equivalent quantity that we will call the global CMC energy defined as

$$E_{CMC} = \frac{1}{4\pi G\mathcal{H}}(\mathcal{V} - \mathcal{V}_{inf}). \tag{9}$$

Rather remarkably, the CMC energy is shown to express the full ADM energy of the time-asymptotic evolution only in terms of the total volume, the Hubble parameter and the topological invariant $\mathcal{V}_{inf}$.

The contents and sections are organized as follows. In section 2 we introduce the main equations for the Einstein-CMC flow as well as Bel-Robinson energies and their main formulae. In section 3 we introduce the averaged cosmological parameters and the Friedman-Lemaître equations. The treatment has no restriction on the sort of matter. We introduce the Newtonian gravitational potential ϕ, its

Poisson-like equation and reformulate the FL equation with it in subsection 3.2. As it turns out the Newtonian potential is the main field to estimate when the purpose is to estimate the universe deceleration and the Hubble parameter as a function of red shift z. In subsection 3.3 we introduce the CMC global energy and relate it in subsection 3.4 with the ADM energy in the weak field limit, analysis extended in subsection 3.5 to arbitrary solutions under assumption (C). In subsection 3.6 we discuss the averaging problem on the light of assumption (C) for the mass gap regime. We will use the CMC energy to estimate the gravitational energy in section 4. Also in section 4 we prove the main estimates of theorem 1. The technique may be thought as estimating the gravitational field through a Taylor expansion (in time) of the zero-order Bel-Robinson tensor and is a natural extension of the analysis in ([6]). In section 5 we construct "big-bang" states of high gravitational energy showing that there is no mathematical reason to assume a low gravitational energy at the initial "big-bang" state. The dynamics of those states even in short times is a completely open problem, in particular it is not know whether the initial rate of expansion with respect to proper time is of matter, radiation or of a type like non of both. In section 6 we give an account of the main points of the article.

2 The CMC flow equations and the Bel-Robinson energies.

2.1 The CMC flow.

In this section we consider the formal setup of the Einstein CMC flow equations. A detailed account can be found in [3]. Consider Σ a compact hyperbolic three-manifold. A cosmological solution to the Einstein equations with compact Cauchy surface Σ is formally a Lorentz metric $\mathbf{g}$ on a four manifold of the form $I \times \Sigma$ (where I is a interval) and where the equal time hypersufaces Σ_t are space-like i.e. the induced metric is Riemannian. If the mean extrinsic curvature ($k = tr_g K$) is constant on each slice of the foliation $\{\Sigma_t\}$ then we say that the cosmological solution is in the (temporal) CMC-gauge. When the spatial topology is a hyperbolic manifold the mean curvature k cannot be zero (due to the energy constraint and the fact that Σ doesn't accept metrics of non-negative scalar curvature) and it can be proved to be strictly monotonic over a unique and connected interval. For a three manifold of hyperbolic type in particular it is conjectured that the CMC foliation has a range of k equal to $I = (-\infty, 0)$, i.e. from a "big-bang" when $\mathcal{H} \rightarrow \infty$ towards an infinitely expanding universe when $\mathcal{H} \rightarrow 0$. Say $\partial_t = NT + X$ where T is the unit normal to the slices and $t = k$. Write the four metric as

$$\mathbf{g} = -N^2 dt^2 + X^* \otimes dt + dt \otimes X^* + g, \tag{10}$$

where g is the spatial three dimensional metric. N is called the lapse and measures the rate of proper time with k (locally). X is called the shift vector field and can be chosen freely but compatible with the regularity. For a discussion of the initial value formulation in the CMC gauge we refer the reader

to [3]. We call the path $(g, N, X)(k)$ the CMC flow. A CMC state is a pair position-normal velocity (g, K) (where K is the second fundamental form and is equal to $K = -\frac{1}{2}\mathcal{L}_T g$) with $k = tr_g K$ constant. Thus the CMC flow gives rise to a flow of position and velocities $(g, K)(k)$. With this notation the Einstein equations

$$\mathbf{Ricc} - \frac{1}{2}\mathbf{R}\mathbf{g} = 8\pi G\mathbf{T}, \tag{11}$$

can be seen as the CMC flow equations (taking $t = k$)

1. *Hamilton-Jacobi equations*

$$g' = -2NK + \mathcal{L}_X g, \tag{12}$$

$$K' = -\nabla^2 N + N(Ricc + kK - 2K \circ K) + \mathcal{L}_X K - 8\pi GN(\mathbf{T} - \frac{tr_{\mathbf{g}}\mathbf{T}}{2}\mathbf{g}), \tag{13}$$

2. *Constraint equations (energy and momentum respectively)*

$$R - |K|^2 + k^2 = 16\pi G\rho, \tag{14}$$

$$\nabla.K = -8\pi GJ, \tag{15}$$

3. *Lapse equation* (deduced from equations above)

$$-\Delta N + (4\pi G(\rho + 3p) + |K|^2)N = 1. \tag{16}$$

The $\mathbf{T}$-term in the right hand side of equation 13 must be thought to be restricted to Σ. Also as usual $\rho = \mathbf{T}(T, T)$, $J = \mathbf{T}(T, .)$ and $p = \frac{(\mathbf{T}_{ab})(g^{ab})}{3}$ is the average of the principal pressures. In equation 15, $\nabla.K = \nabla^a K_{ab}$ is the divergence and in equation 13 it is $(K \circ K)_{ab} = K_{ac}K^c{}_b$. Finally the speed of light was taken to be $c = 1$.

2.2 The Bel-Robinson energy and the space time curvature.

We will measure the L^2 norm of the space time curvature relative to the CMC gauge. We will also need to measure the L^2 norm of their time derivatives relative to the normal direction to the CMC foliation. There is a remarkable way to introduce them and it is by means of Weyl fields. Although we won't discuss Weyl fields in detail as there are very accurate references on the subject ([7],[6]), we will mention the most used properties here and briefly elaborate on their conceptual importance as variables controlling the gravitational field.

Definition 1 *A Weyl field is a traceless* $(4, 0)$ *space time tensor satisfying the symmetries of the curvature tensor* $\mathbf{Rm}$*. We will denote them by* $\mathbf{W}_{abcd}$ *or simply* $\mathbf{W}$*.*

The Riemann tensor of a vacuum solution to the Einstein equations is a Weyl field that we will denote as $\mathbf{Rm} = \mathbf{W}_0$. Let T be the normal field (future pointing) to the CMC foliation. Then $\boldsymbol{\nabla}_T^i \mathbf{W}_0 = \mathbf{W}_i$ are Weyl fields. Together with the *volume radius* ([3]) and the L^2 norm of the second fundamental form K they are an important set of variables that control the gravitational field (i.e. the metric $\mathbf{g}$ relative to the foliation) see [3]. A central advantage for taking them as variables is that they enjoy remarkable algebraic properties that simplifies the space time algebra considerably. We discuss the main formulae below. Given a Weyl tensor $\mathbf{W}$ define the left and right duals ${}^*\mathbf{W}_{abcd} = \frac{1}{2}\epsilon_{ablm}\mathbf{W}^{lm}{}_{cd}$ and $\mathbf{W}^*_{abcd} = \mathbf{W}_{ab}{}^{lm}\frac{1}{2}\epsilon_{lmcd}$. Both are Weyl tensors, ${}^*\mathbf{W} = \mathbf{W}^*$ and ${}^*({}^*\mathbf{W}) = -\mathbf{W}$. Define the current $J(\mathbf{W})$ and its dual $J^*(\mathbf{W})$ as

$$\boldsymbol{\nabla}^a \mathbf{W}_{abcd} = J_{abc}(\mathbf{W}), \tag{17}$$

$$\boldsymbol{\nabla}^a \mathbf{W}^*{}_{abcd} = J^*_{abc}(\mathbf{W}). \tag{18}$$

For the Riemann tensor in a vacuum solution to the Einstein equation we have $J = J^* = 0$ due to the Bianchi equations. This is a central fact that will be of fundamental importance latter. We also have

$$\boldsymbol{\nabla}_{[a}\mathbf{W}_{bc]de} = \tfrac{1}{3}\epsilon_{fabc}J^{*f}_{de}(\mathbf{W}), \tag{19}$$

$$\nabla_{[a}\mathbf{W}^*_{bc]}de = \frac{1}{3}\epsilon_{fabc}J^f_{de}(\mathbf{W}). \tag{20}$$

The L^2 norm with respect to the foliation will be defined through the Bel-Robinson tensor. Given a Weyl field $\mathbf{W}$ its Bel-Robinson tensor is

$$Q_{abcd}(\mathbf{W}) = \mathbf{W}_{alcm}\mathbf{W}_b{}^l{}_d{}^m + \mathbf{W}^*_{alcm}\mathbf{W}^*{}_b{}^l{}_d{}^m. \tag{21}$$

It is symmetric and traceless in all pair of indices and for any pair of timelike vectors T_1 and T_2, $Q(T_1, T_1, T_2, T_2)$ is positive if $\mathbf{W} \neq 0$ ([7]). In particular we define the L^2 norm of $\mathbf{W}$ with respect to the foliation as $Q(T, T, T, T)$. It is seen to be the L^2 norm of the electric and magnetic fields of $\mathbf{W}$ defined through

$$E_{ab}(\mathbf{W}) = \mathbf{W}_{acbd}T^cT^d, \tag{22}$$

$$B_{ab}(\mathbf{W}) =^* \mathbf{W}_{acbd}T^cT^d. \tag{23}$$

i.e. $Q(T, T, T, T) = |E|^2 + |B|^2$. They are symmetric, traceless and null on the T direction. For the Riemann tensor in particular we have

$$E_{ab}(\mathbf{W}_0) = Ricc_{ab} + kK_{ab} - Ka\ {}^cK^c{}_b, \tag{24}$$

and

$$\epsilon_{ab}{}^l B_{lc}(\mathbf{W}_0) = \nabla_a K_{bc} - \nabla_b K_{ac}. \tag{25}$$

The following formulae provide the components of a Weyl field with respect to the CMC foliation in terms of the electric and magnetic fields (i, j, k, l are spatial indices)

$$\mathbf{W}_{ijkT} = -\epsilon_{ij}{}^{m} B_{mk}(\mathbf{W}),\ {}^{*}\mathbf{W}_{ijkT} = \epsilon_{ij}{}^{m} E_{mk}(\mathbf{W}), \tag{26}$$

$$\mathbf{W}_{ijkl} = \epsilon_{ijm}\epsilon_{kln} E^{mn}(\mathbf{W}),\ {}^{*}\mathbf{W}_{ijkl} = \epsilon_{ijm}\epsilon_{kln} B^{mn}(\mathbf{W}). \tag{27}$$

The divergence formula

$$\boldsymbol{\nabla}^{a} Q(\mathbf{W})_{abcd} = \mathbf{W}_{b\ d}^{\ m\ \ n} J(\mathbf{W})_{mcn} + \mathbf{W}_{b\ c}^{\ m\ \ n} J(\mathbf{W})_{mdn} + \tag{28}$$

$$+^{*}\ \mathbf{W}_{b\ d}^{m\ n} J^{*}(\mathbf{W})_{mcn} +^{*}\ \mathbf{W}_{b\ c}^{m\ n} J^{*}(W)_{mcn}, \tag{29}$$

and therefore

$$\boldsymbol{\nabla}^{\alpha} Q(\mathbf{W})_{\alpha TTT} = 2E^{ij}(\mathbf{W}) J(\mathbf{W})_{iTj} + 2B^{ij} J^{*}(\mathbf{W})_{iTj} \tag{30}$$

gives the *Gauss equation*

$$\frac{\partial \int_{\Sigma} Q(T,T,T,T) dv_{g}}{\partial t} = -\int_{\Sigma} 2NE^{ij}(\mathbf{W}) J(\mathbf{W})_{iTj} + 2NB^{ij} J^{*}(\mathbf{W})_{iTj} + 3NQ_{abTT}\Pi^{ab} dv_{g}, \tag{31}$$

where $\Pi_{ab} = \boldsymbol{\nabla}_{a} T_{b}$ is the deformation tensor and plays a fundamental role in the tensor algebra. In terms of the electric and magnetic fields the components of Q_{abTT} are written as

$$Q_{iTTT} = 2(E \wedge B)_{i}, \tag{32}$$

$$Q_{ijTT} = -(E \times E)_{ij} - (B \times B)_{ij} + \frac{1}{3}(|E|^{2} + |B|^{2}) g_{ij}. \tag{33}$$

Controlling J and J^{*} in L^{2} and Π in H^{2} is enough to control the L^{2} norm of the Weyl field. The following formulas are essential when it comes to get Sobolev estimates of the Weyl field

$$divE(\mathbf{W})_{a} = (K \wedge B(\mathbf{W}))_{a} + J_{TaT}(\mathbf{W}), \tag{34}$$

$$divB(\mathbf{W})_{a} = -(K \wedge E(\mathbf{W})) + J^{*}_{TaT}(\mathbf{W}), \tag{35}$$

$$curlB_{ab}(\mathbf{W}) = E(\boldsymbol{\nabla}_{T}\mathbf{W})_{ab} + \tfrac{3}{2}(E(\mathbf{W}) \times K)_{ab} - \tfrac{1}{2} k E_{ab}(\mathbf{W}) + J_{aTb}(\mathbf{W}), \tag{36}$$

$$curlE_{ab}(\mathbf{W}) = B(\boldsymbol{\nabla}_{T}\mathbf{W})_{ab} + \tfrac{3}{2}(B(\mathbf{W}) \times K)_{ab} - \tfrac{1}{2} k B(\mathbf{W})_{ab} + J^{*}_{aTb}(\mathbf{W}), \tag{37}$$

where the operations $\wedge,\ \times$ are defined as

$$(A \times B)_{ab} = \epsilon_{a}{}^{cd} \epsilon_{b}{}^{ef} A_{ce} B_{df} + \frac{1}{3}(A - B) g_{ab} - \frac{1}{3}(trA)(trB) g_{ab}, \tag{38}$$

$$(A \wedge B)_{a} = \epsilon_{a}{}^{bc} A_{b}{}^{d} B_{dc}. \tag{39}$$

The equations 34-37 above are an example of the so called elliptic Hodge systems ([7]). In particular under basic regularity of the background metric they make possible to get elliptic estimates.

2.3 Scaling.

Scaling is the operation allowing us to speak like "looking the system at a particular scale". It is a different operation of than coordinate scaling, as scaling a solution does changes the solution but scaling coordinate systems doesn't. Both transformations are however important when used simultaneously.

Definition 2 *Given a solution* $\mathbf{g}$ *to the Einstein equations, we call* $\lambda^2\mathbf{g}$ *the solution* $\mathbf{g}$ *at the scale of* $\frac{1}{\lambda}$ *and we call* λ *the scale factor.*

We say that a CMC state (g, K) is cosmologically scaled (or normalized) if $k = -3$ or the same $\mathcal{H} = 1$ as we will see the Hubble parameter $\mathcal{H}$ is equal to $\frac{-k}{3}$. Given a state (g, K) that gives rise to a global solution $\mathbf{g}$ we can scale it as $\frac{k^2}{9}\mathbf{g}$ to transform the original state $(g, K)(k)$ into a cosmologically normalized state $(\frac{k^2}{9}g, \frac{-k}{3}K)$. Therefore a state (g, K) has a cosmological scale of $\frac{3}{-k} = \frac{1}{\mathcal{H}}$. Say $(g, K)(k)$ is a CMC state, and say U is some space time tensor constructed out of $\mathbf{g}$ that we are looking at the k-slice. The corresponding values of U on the same slice when we cosmologically scale the state (g, K) will be denoted with a tilde (either above or next to it) say $\tilde{U}$ or $U^{\sim}$. Thus $\tilde{g} = \frac{k^2}{9}g$ and $\tilde{K} = \frac{-k}{3}K$. In a CMC flow $(g, K)(k)$ we can cosmologically scale the solution $\mathbf{g}$ at every k getting thus a flow of normalized states $(\tilde{g}, \tilde{K})(k)$. In the flat cone case the cosmologically scaled flow is just $(g_H, -g_H)(k)$ and what we will call stability of the flat cone will be the stability of the cosmologically scaled solutions. In general a space time tensor will scale as $\lambda^s U$ for some weight s, therefore $\tilde{U}$ will be just $\tilde{U} = (\frac{-k}{3})^s U$. We will indistinctly use $\frac{-k}{3}$ or $\mathcal{H}$ as the scale factor λ. The following table shows how some main tensors transform when $\mathbf{g} \to \lambda^2\mathbf{g}$.

$\mathbf{g}$	$\lambda^2\mathbf{g}$
g	$\lambda^2 g$
K	λK
k	$\frac{k}{\lambda}$
N	$\lambda^2 N$
ϕ	ϕ
$\mathbf{W}_i$	$\lambda^{-i+2}\mathbf{W}_i$
Q_i	$\lambda^{-(2i+1)}Q_i$

ϕ is the Newtonian potential defined below.

3 Averaged evolution.

3.1 Averaged cosmological parameters and the averaging map.

We define the geometric parameters, $a(k)$ (universe's radius), $\tau(k)$ (proper time), and $\mathcal{H}(k)$ (Hubble parameter) in volume average. All those parameters reduce to the standard FL parameters when the solution is homogeneous and isotropic.

Definition 3 *Given an arbitrary CMC solution we define the universe's radius at an instant of time k as $a(k) = V_{g(k)}^{\frac{1}{3}}$. The volume-averaged proper time $\tau(k)$ is defined through*

$$\frac{d\tau}{dk} = \frac{\int_\Sigma N dv_g}{V}. \tag{40}$$

Recalling that in the FL models the Hubble parameter is defined as $\mathcal{H} = \frac{1}{a}\frac{da}{d\tau}$ we compute

$$\mathcal{H} = \frac{1}{V^{\frac{1}{3}}}\frac{dV^{\frac{1}{3}}}{d\tau} = \frac{1}{V^{\frac{1}{3}}}\frac{dV^{\frac{1}{3}}}{dk}\frac{dk}{d\tau} = \frac{1}{V^{\frac{1}{3}}}\frac{1}{3}V^{-\frac{2}{3}}(\int_\Sigma -Nkdv_g)\frac{V}{\int_\Sigma N dv_g} = \frac{-k}{3}. \tag{41}$$

Thus in arbitrary solutions $\mathcal{H} = \frac{-k}{3}$. This expression is valid also locally in the following sense: define the cube of the local radius as the volume element $dv_g(k)$, then the local Hubble parameter is one third the logarithmic derivative of the volume element with respect to the proper time in the normal direction to the CMC slice k. A direct computation gives for the local Hubble parameter $\mathcal{H} = \frac{1}{3dv_g}\frac{dv_g}{d\tau} = \frac{-k}{3}$.

The Friedman-Lemaître equations take the form

1. First FL equation : $$\mathcal{H}^2 = -\frac{\int_\Sigma R dv_g}{6V} + \frac{\int_\Sigma (16\pi G\rho + |\hat{K}|^2) dv_g}{6V}, \tag{42}$$

2. Second FL equation : $$\frac{a''}{a} = \frac{-\int_\Sigma \mathcal{N}(4\pi G(\rho + 3p) + |\hat{K}|^2) dv_g}{3V}. \tag{43}$$

Where $\mathcal{N} = \frac{N}{\bar{N}}$ (bar denotes volume-average) and has average equal to one. The derivatives, denoted with a prime are proper time derivatives, i.e. $' = \frac{d}{d\tau}$. The first FL equation is just the volume average of the energy constraint

$$16\pi\rho = R - |\hat{K}|^2 + \frac{2}{3}k^2. \tag{44}$$

Observe that to make it look closer to the second FL equation, we can multiply the energy constraint before integrating by $\mathcal{N}$ and integrate thereafter to get

$$\mathcal{H}^2 = \frac{\int_\Sigma \mathcal{N} R dv_g}{6V} + \frac{\int_\Sigma \mathcal{N}(16\pi G\rho + |\hat{K}|^2) dv_g}{6V}. \tag{45}$$

To obtain the second FL equation we observe that

$$(\frac{a'}{a})' = \frac{a''}{a} - (\frac{a'}{a})^2 = \frac{a''}{a} - \mathcal{H}^2, \tag{46}$$

and

$$(\frac{a'}{a})' = \frac{d\mathcal{H}}{d\tau} = -\frac{1}{3}\frac{dk}{d\tau} = -\frac{V}{3\int_\Sigma N dv_g}. \tag{47}$$

On the other hand integrating the Lapse equation 16 we get

$$\int_\Sigma N(4\pi G(\rho + 3p) + |\hat{K}|^2) dv_g = V - 3\mathcal{H}^2 \int_\Sigma N dv_g. \tag{48}$$

Equations 46, 47, 48 together give equation 43.

Let's restate the standard $\mathcal{K} = -1$ FL models on the light of the description given above for arbitrary solutions. If the solution is $\mathbf{g} = -d\tau^2 + a(\tau)^2 g_H$ on a manifold $\mathbb{R} \times \Sigma$ then $a(\tau) = (\frac{V}{V_H})^{\frac{1}{3}}$ where V_H is the volume of Σ with the hyperbolic metric g_H and V is the volume with the metric $a(\tau)^2 g_H$. Our choice of radius for arbitrary solutions has been instead $a(\tau) = V^{\frac{1}{3}}$, we will make this choice in equations 49 and 50 below. We recall too that in the standard FL models the energy density and pressures are a function only of τ and for that reason they coincide with their volume averages. Taking these facts into account the standard FL equations are

1. $$\mathcal{H}^2 = \frac{\int_\Sigma (16\pi G\rho) dv_g}{6V} - \frac{\mathcal{K} V_H^{\frac{2}{3}}}{a^2}. \tag{49}$$
2. $$\frac{a''}{a} = \frac{-\int_\Sigma (4\pi G(\rho + 3p) dv_g}{3V}. \tag{50}$$

Observe that in the FL equation 42 instead of the curvature term $-\mathcal{K} V_H^{\frac{2}{3}}/a^2$ we have the term

$$-\frac{\int_\Sigma R dv_g}{6V} = -(\frac{\int_\Sigma R dv_g}{6V^{\frac{1}{3}}})\frac{1}{a^2}, \tag{51}$$

where the first factor in the last term of the previous equations is scale invariant and therefore equal to $V_H^{\frac{2}{3}}$ for any metric scaled from the hyperbolic metric (so is close to it for any metric scaled from a metric close to a hyperbolic metric).

In order to establish a mathematical definition of the averaging problem in cosmology we define the *averaging map* from arbitrary CMC solutions into Lorentzian manifolds in the following way.

Definition 4 *Given an arbitrary CMC solution* $\mathbf{g}$ *on* $\mathbb{R} \times \Sigma$ *with* Σ *a compact hyperbolic manifold define the volume-averaged solution as the Lorentzian space* $(\mathbb{R} \times \Sigma, \mathcal{A}(\mathbf{g}))$ *with* $\mathcal{A}(\mathbf{g}) = \bar{\mathbf{g}} = -d\tau^2 + \frac{a(\tau)^2}{V_H^{\frac{2}{3}}} g_H$*, where* τ *and* $a(\tau)$ *are the averaged proper time and radius as given in definition 3.* g_H *is the unique (up to diffeomorphism) hyperbolic metric that* Σ *accepts.*

It is essential in the definition above that, due to Mostow's rigidity, there is one hyperbolic metric up to diffeomorphism in a given hyperbolic manifold. That makes the definition of $\mathcal{A}(\mathbf{g})$ unambiguous.

In rough terms the averaging problem for arbitrary solutions can be stated as to whether the averaged space $\mathcal{A}(\mathbf{g})$ is "asymptotically in time close" to an exact $\mathcal{K} = -1$ FL solution with the

"averaged energy density and pressures" "asymptotically in time close" to the energy density and pressures of the exact FL model. One may also replace "asymptotically in time close" simply by "close" all along evolution. Physically that would be a more adequate question to ask. This definition however faces various indefiniteness, we comment on them below.

i) The first is to give a precise meaning to "averaged energy and pressures" for arbitrary solutions. We can safely say what they are for the material fields, as material fields posses densities of pressures and energy, but it is not known what they are for the gravitational field, and presumably they can not be isolated as densities. The old question on how to define the gravitational energy which shows up all through General Relativity is present also here. A consensual definition of energy is the total ADM energy, a global term comprising the energetic content of a global system. Despite all the satisfactory the expression is, it is defined in asymptotically flat space-times and not in the context of cosmological solutions. We will argue in subsection 3.6 on the validity of the averaging problem , at least asymptotically in time, if it is assumed a compact and extended relative of the *weak cosmic censorship conjecture* of Penrose, conjecture stated for asymptotically flat space-times. Indeed we will analyze the averaging problem under the assumption that, under a particular model for matter at natural scales (the small structure), it happens that, generically, cosmological solutions evolve into a finite set (however large it may be) of asymptotically flat stationary solutions separating from each other, with gravitational radiation in between and if in addition we compute the "averaged energy density" as the volume-average of the ADM energies of the stationary solutions plus the volume-average energy of the gravitational radiation in between. Both terms, as we shall see, can safely be computed. We will call the assumption above *assumption* (C). The extent as to whether this idealized assumption would be applicable to the actual universe in which we live at present times is not under consideration here. However I would like to point out one aspect that immediately jumps out and that it would have to be addressed with care. Assuming that galaxies conform the individual stationary solutions, there is the issue to establish, due to the large dark halos extended over diameters many times their visible diameters, where (if somewhere) and how far the individual galaxies (including their halos) become asymptotically flat. This lack of asymptotic flatness on large neighborhoods around the visible galaxy is manifested in the well known flat rotation curves of stars with large orbital radius.

ii) A second problem in the rough definition of the averaging problem given above is to specify the equation of state of the exact FL solution from the original solution at natural scales. On the light of assumption (C) there are two situations possible, a radiative regime, of universes filled only with gravitational radiation, and a massive regime, of universes where in addition to radiation there are massive compact object (the stationary solutions). Both regimes, that we will call *radiation* and *mass gap* respectively, deserve different technical analysis. We will discuss the radiation regime in rigor and detail below in section 4. The analysis of the mass gap regime is done in subsection 3.5. Although rigorously deduced from the assumption (C), it lacks a precise determination on the decay

of the radiation term. We will return to this point later.

iii) A third problem is to define in a quantitative manner the notion of "closeness" between the averaged space and the exact FL solution. Precisely, we have to specify the scale in which the solutions are compared and a law for the asymptotic relation between them.

3.2 The Friedman-Lemaître equations and the Newtonian potential.

A remarkable fact about the averaging formalism is that the second FL equation can be written only in terms of the volume-average of the Newtonian potential $\bar{\phi}$ and consequently $a(\tau)$, $\mathcal{H}(\tau)$ and $z(\tau)$ are determined only from $\bar{\phi}$.

DEFINITION 5 *Define the Newtonian potential ϕ as $\phi = \frac{Nk^2}{3} - 1$. It satisfies the Poisson equation (Lapse equation)*

$$\Delta\phi = (4\pi G(\rho + 3p) + |\hat{K}|^2) + (4\pi G(\rho + 3p) + |\hat{K}|^2 + 3H^2)\phi, \tag{52}$$

or making $e = 4\pi G(\rho + 3p) + |\hat{K}|^2$

$$\Delta\phi = e + (e + 3H^2)\phi. \tag{53}$$

From the Maximum principle it is seen that $-1 \leq \phi \leq 0$. Observe too that ϕ is an absolute potential, i.e. there is no ambiguity in the level of energy in its definition (as can be deduced from the unicity of solutions in equation 52) and observe also that it is scale invariant. As defined here the Newtonian potential of course coincides with the usual Newtonian potential in the weak field Newtonian regime (when $p \approx 0$ and $K \approx 0$). Compare also equation 52 with the usual Poisson equation in Newtonian dynamics

$$\Delta\phi = 4\pi G\rho. \tag{54}$$

Equation 52 is fundamental to understand the dynamics of the gravitational field in general and its analysis extracts among other things the time at which Newtonian dynamics appears, i.e. when is that gravitation gets ruled by classical Newtonian potentials at large scales. A straightforward calculation gives

$$\frac{a''}{a} = \mathcal{H}^2 \frac{\bar{\phi}}{1 + \bar{\phi}}, \tag{55}$$

or

$$\frac{\mathcal{H}'}{\mathcal{H}^2} = \frac{-1}{1 + \bar{\phi}}. \tag{56}$$

where $\bar{\phi}$ is the volume-average of ϕ. This equation can be used to get an equation for $\mathcal{H}$ as a function of red shift $1+z=\frac{V^{\frac{1}{3}}}{V(z)^{\frac{1}{3}}}$ (V is the present volume and $V(z)$ is the volume at the corresponding red shift). The relation is

$$\frac{d\ln\mathcal{H}}{d\ln(1+z)}=\frac{1}{1+\bar{\phi}}. \tag{57}$$

One also obtains

$$\frac{d\ln(1+z)}{d\tau}=-\mathcal{H}. \tag{58}$$

Of course an estimation of $\bar{\phi}$ as a function of τ, z or $\mathcal{H}$ is needed to make use of the equations above.

3.3 The CMC energy.

We would like to define a formal quantity on CMC states on a compact manifold Σ analogous to the total ADM mass of asymptotically flat space-times. Restate the first FL equation 49 in the form

$$1-(\frac{\mathcal{V}_{inf}}{\mathcal{V}})^{\frac{2}{3}}=\Omega_m, \tag{59}$$

where we have defined $\mathcal{V}_{inf}$ as the absolute infimum of the reduced volume $\mathcal{V}=\mathcal{H}^3V(g,K)$ among the set of all CMC states (g,K). It is known [9],[3] that if Σ is hyperbolic $\mathcal{V}_{inf}=V_H$. Ω_m is defined as usual as $\Omega_m=\frac{8\pi G\rho}{3\mathcal{H}^2}$. Thus the density of mass ρ and the Hubble parameter $\mathcal{H}$ determine the deviation of the reduced volume from its absolute infimum. If $\frac{8\pi G\rho}{3H^2}\sim 0$ we get in particular the approximation

$$\mathcal{M}\approx\frac{1}{4\pi G\mathcal{H}}(\mathcal{V}-\mathcal{V}_{inf}). \tag{60}$$

This remarkable equation, expresses the total mass $\mathcal{M}$ in terms only of $\mathcal{H}$, G, the total volume V and the topological invariant V_H. As we shall see in section 3.5 it holds too, asymptotically in time, for general models under assumption (C). Inspired on it and equation 60 we define the total CMC energy as

Definition 6 *Define the CMC global energy as*

$$E_{CMC}=\frac{1}{4\pi G\mathcal{H}}(\mathcal{V}-\mathcal{V}_{inf}). \tag{61}$$

3.4 The ADM limit of the CMC energy: radiation.

Recall that the Hessian of the ADM energy around the flat Minkowski space-time state $g=g_E$ and $K=0$ (g_E is the euclidean metric) is (see for instance [8])

$$8\pi G\delta^{(2)}E_{ADM}=\frac{1}{4}\int_{\mathbb{R}^3}|\nabla g'_{TT}|^2dv+\int_{\mathbb{R}^3}|K'_{TT}|^2dv+8\pi G\int_{\mathbb{R}^3}\delta^{(2)}\rho dv, \tag{62}$$

where TT means transverse-traceless with respect to the flat metric g_E. The Hessian of the reduced volume $\mathcal{V}$ was calculated in [9]. We include below a calculation of the Hessian of the CMC energy 61 based on their analysis for the sake of completeness and clarity. The Hessian of the CMC energy in the limit when $k \to 0$ is locally the same as equation 62, the precise expression is

$$8\pi G\delta^{(2)}E_{CMC} = \int_\Sigma |K'_{TT}|^2 dv_g + \frac{1}{4}\int_\Sigma |\nabla g'_{TT}|^2 dv_g - \frac{\mathcal{H}^2}{2}\int_\Sigma |g'_{TT}|^2 dv_g + 8\pi G\int_\Sigma \delta^{(2)}\rho dv_g. \quad (63)$$

where the background state is $(\frac{9}{k^2}g_H, \frac{3}{k}g_H)$. We thus see the local vanishing of the third term on the right hand side when $\mathcal{H} \to 0$. Observe that the kinetic term $\frac{|\hat{K}|^2}{16\pi G}$ deduced from expression 63 (there is an extra factor of a half when we read the energy from its Hessian) is consistent with the first and second FL equations in the radiation regime, where the densities of gravitational energy and pressure, are unequivocally identified with $\rho_G = p_G = \frac{|\hat{K}|^2}{16\pi G}$. Note however that the first term in 63 doesn't form part of the effective densities of gravitational energy and pressure in equation 47 and therefore doesn't influence the universe's deceleration, instead it is part of the curvature term in the first FL equation.

The calculation of the Hessian is as follows. In terms of conformal variables, a state (g, K) is written as

$$g_{ab} = \varphi^4 g_{Y,ab}, \quad (64)$$

$$K^{ab} = \varphi^{-10}\hat{K}_Y^{ab} + \frac{k}{3}\varphi^{-4}g_Y^{ab}. \quad (65)$$

Where g_Y is a Yamabe metric of constant scalar curvature $R_Y = -6\frac{k^2}{9}$ and $\hat{K}_Y$ is a transverse traceless tensor with respect to g_Y. The conformal factor φ must satisfy the Lichnerowicz equation

$$\Delta\varphi + \frac{k^2}{12}(\varphi - \varphi^5) + \frac{|\hat{K}_Y|_Y^2}{8}\varphi^{-7} + 2\pi G\rho\varphi^5 = 0. \quad (66)$$

We will take derivatives along a path $(g, K)(\lambda)$ with $(g, K)(0) = (\frac{9}{k^2}g_H, \frac{3}{k}g_H)$, which in turn can be seen as a path $(g_Y, K_Y, \varphi)(\lambda)$. Note that $\varphi(0) = 1$. Recalling the derivative of the Laplacian ([10])

$$-(\Delta')f =< \nabla^2 f, g' > - < \nabla f, \delta h > -\frac{1}{2} < \nabla f, dtr_g g' >, \quad (67)$$

the first derivative at $\lambda = 0$ of the Lichnerowicz equation is (we are assuming $\delta^{(1)}\rho = 0$)

$$\Delta\varphi' - \frac{k^2}{3}\varphi' = 0. \quad (68)$$

which shows that $\varphi'(0) = 0$ identically. Using that fact we get

$$V''(0) = (\int_\Sigma \varphi^6 dv_g)'' = 6\int_\Sigma \varphi'' dv_{g(0)} + \int_\Sigma dv''_{g_Y}. \quad (69)$$

Integrating the Lichnerowicz equation and differentiating the integral equation twice gives

$$\frac{8k^2}{3}\int_\Sigma \varphi'' dv_{g(0)} = 2\int_\Sigma |\hat{K}'_Y|^2 dv_{g(0)} + 16\pi G\int_\Sigma \delta^{(2)}\rho dv_{g(0)}, \quad (70)$$

from which we get

$$6\int_{\Sigma}\varphi'' dv_{g(0)} = \frac{9}{2k^2}\int_{\Sigma}|\hat{K}'|^2 dv_{g(0)} + \frac{9}{2k^2}8\pi G\int_{\Sigma}\delta^{(2)}\rho dv_{g(0)}. \tag{71}$$

Now let's compute the second term in equation 69. First we note that

$$dv''_{g_Y} = (\frac{tr_{g_Y} g''_Y}{2} - \frac{|g'_Y|^2}{2} + (\frac{tr_g g'}{2})^2) dv_{g_Y}. \tag{72}$$

To compute $tr_{g_Y} g''_Y$ we will use the variation formula for the scalar curvature. As the metrics g_Y are Yamabe of scalar curvature $-6\frac{k^2}{9}$ the derivative in λ of R_Y is zero pointwise, precisely ([10])

$$R' = -\Delta(tr_{g_Y} g'_Y) + \delta\delta g'_Y - < Ric, g' > = 0. \tag{73}$$

Integrating we get

$$\int_{\Sigma} < Ric, g'_Y > dv_{g_Y} = 0, \tag{74}$$

for all λ. Differentiating again at $\lambda = 0$ we get

$$\int_{\Sigma}(< Ric', g'_Y > + < Ric, g''_Y > + (Ric_{ab})(g'_{Y,cd})(g^{ac}_Y)'(g^{bd}_Y) + (Ric_{ab})(g'_{Y,cd})(g^{ac}_Y)(g^{bd}_Y)') dv_{g(0)} \tag{75}$$

The Ricci curvature at $\lambda = 0$ is $Ric = \frac{-2k^2}{9} g_H$. Also the functional derivative of Ricci is

$$Ric' = \frac{1}{2}\Delta_L g' - \delta^*(\delta g') - \frac{1}{2}\nabla\nabla(tr_g g'). \tag{76}$$

Observe that from equation 73 we have $tr_{g(0)} g'(0) = 0$. Δ_L is the Lichnerowicz laplacian and has the expression [10]

$$\Delta_L T_{ab} = \nabla^*\nabla T_{ab} + (Ric_{ac}T^c_{\ b} + Ric_{bc}T^c_{\ a}) - (Rm_{acbd}T^{cd} + Rm_{bcad}T^{dc}). \tag{77}$$

Using both facts and also that g' is taken to be transverse we get from equation 75 that

$$\int_{\Sigma} tr_{g_Y} g''_Y dv_{g(0)} = 2\int_{\Sigma}|g'_Y|^2 dv_{g(0)} + \frac{9}{4k^2}\int_{\Sigma} < g'_y, \Delta_L g'_Y > dv_{g(0)}. \tag{78}$$

To compute the Lichnerowicz laplacian we remember that the sectional curvature of $g(0)$ is $-\frac{k^2}{9}$, therefore

$$\Delta_L g'_Y = \nabla^*\nabla g'_Y - \frac{6k^2}{9} g'_Y. \tag{79}$$

at $\lambda = 0$. Using the previous equation in equation 78, we get the result of equation 63 after putting together equations 72, 70, 69.

3.5 The long time ADM limit of the CMC energy: radiation and mass gap.

In this subsection we will introduce assumption (C) and show how, under that assumption, the CMC energy converges assymptotically in time to the sum of the ADM masses of the emerging stationary solutions plus a radiative term of the radiation in between. The analysis will lead us to argue in subsection 3.6 on the validity of the averaging problem in cosmology under assumption (C) and asymptotically in time. First we recall the definition of asymptotically flat stationary solution.

DEFINITION 7 *([11], pg 16) A maximal ($k = 0$) initial data set (g, K, N, X) is a stationary asymptotically flat data iff*

1. *(a)* $g_{00} = -(1 - \frac{2M}{r}) + O(r^{-2})$,

   (b) $g_{ij} = (1 + O(r^{-1}))\delta_{ij} + O(r^{-2})$,

   (c) $g_{0i} = -\epsilon_{ijk}\frac{4S^j}{r^3}x^k + O(r^{-3})$.

2. *it satisfies the stationary vacuum Einstein equations* $\dot{g} = \dot{K} = 0$.

Now we state the definition of assumption (C). A schematic representation of a space-time (at a given time) satisfying assumption (C) is given in figure 1.

DEFINITION 8 *A long time CMC solution satisfies the assumption (C) iff:*

1. *(emergence of isolated stationary solutions) after a sufficiently large time there is a finite set of pairs of two-spheres (inner and outer) with constant mean curvatures* $2/L_0$ *and* $2/L(t)$ *respectively, varying continuously in time* ($t = 1/\mathcal{H}$) *such that, inside the annulus in between, the unscaled flow* (g, K, N, X) *decays in the* C^1 *norm into a stationary solution* (g_0, K_0, N_0, X_0)*. At the outer spheres,* $|\nabla\phi - \nabla\phi_0| \leq \frac{C}{L(t)^2 t^{1+\epsilon}}$.

2. *(the inside of the inner spheres) after a sufficiently long time the volume of the inside of the inner spheres grows no faster than* $t^{1-\epsilon}$.

3. *(emergence of the radiative region) after a sufficiently long time the cosmologically normalized flow* $(\tilde{g}, \tilde{K}, \tilde{N}, \tilde{X})$*, decays uniformly in* C^1*, over the exterior region to the outer spheres into the flat cone state* $(g_H, -g_H, 1/3, 0)$.

4. *(boundedness of the CMC energy)* $\frac{dE}{dt} \leq \frac{C}{t^{2+\epsilon}} \to 0$.

Some remarks are in order. The interior radius L_0 is fixed. The exterior radius $L(t)$ grows monotonically but less than t: $\lim_{t\to\infty} \frac{L(t)}{t} = 0$, in such a way that at cosmological scales the outer spheres get smaller and smaller in size. Similarly, the rate at which the solution over the annulus decays into

the stationary solution and the rate at which the solution on the exterior region decays into the flat cone solution are left unspecified here. Item four is a global condition that complements the absence of explicit decaying rates in assumption (C). In the CMC flow, the interior regions of black holes are expected to evolve as tubes of increasingly large size, and therefore increasing volume. Item two gives a bound on its grow in the case they form. We want to stress that all these conditions are tentative and are not intended to be conjectural. Neither we conjecture a sort of assumption (C) to hold generically. The introduction of assumption (C), we believe, provides an starting point in the study of the averaging problem directly from the small structure of exact solutions. All these problems are, however, difficult problems in the field. Section 4 is an attempt to clarify these issues in pure radiative solutions.

Now let's see how the CMC energy behaves under assumption (C). The second FL equation in terms of the CMC energy is

$$\frac{dE_{CMC}}{d\sigma} = -\int_{\Sigma}((\rho+3p)+\frac{|\hat{K}|^2}{4\pi G})(1+\phi)dv_g + E_{CMC} = 3\mathcal{H}^2\int_{\Sigma}\phi dv_g + E_{CMC}, \tag{80}$$

where $\sigma = -\ln -k$ is the logarithmic time. From item four and equation 80 the CMC energy converges to the term $-3\mathcal{H}^2\int_{\Sigma}\phi dv_g$ with a difference bounded by $C/t^{1+\epsilon}$. Now let's separate the region of integration into the inside of the outer spheres and its outside. Using the Poisson equation 52 we get

$$E_{CMC} = \int_{S_{out}} < \nabla\phi, n_{out} > dA + 3\mathcal{H}^2\int_{\Omega_{int}}\phi dv_g + \int_{\Omega_{ext}}((\rho+3p)+\frac{|\hat{K}|^2}{4\pi G})(1+\phi)dv_g + O(t^{-(1+\epsilon)}), \tag{81}$$

where Ω_{int} is the interior of the outer spheres and Ω_{ext} its exterior. Due to item two in assumption (C), the second term on the right hand side of equation 81 is an $O(t^{-(1+\epsilon)})$. The boundary term approach with an error $O(t^{-(1+\epsilon)})$ to the sum of the ADM masses of the emerging stationary solutions. We can identify the third term on the right hand side of equation 81 is the radiative term because by item three $\phi \to 0$ pointwise on Ω_{ext} and the radiation terms from matter and gravitation decouple. Thus we get

$$E_{CMC} \approx \mathcal{M} + \mathcal{R}, \tag{82}$$

the total ADM mass plus the radiation energy. This is the same equation as 60 with the additional radiative term. A remark has to be said about the radiative term. In an asymptotically flat context the ADM energy is a conserved quantity, therefore the radiative contribution to energy measured as the difference between the asymptotically Bondi energy and the ADM energy would be a definite non-zero amount. In other words there is a definite amount of radiative energy that forms part of the ADM energy. In our context, that amount would form part of the radiative term $\mathcal{R}$. Further work is needed to show that, indeed there may exist a non vanishing residual radiative energy in the $\mathcal{R}$ term.

We will use the total CMC energy in section 4 to give a rigorous estimation of the gravitational energy in the long time for radiative solutions.

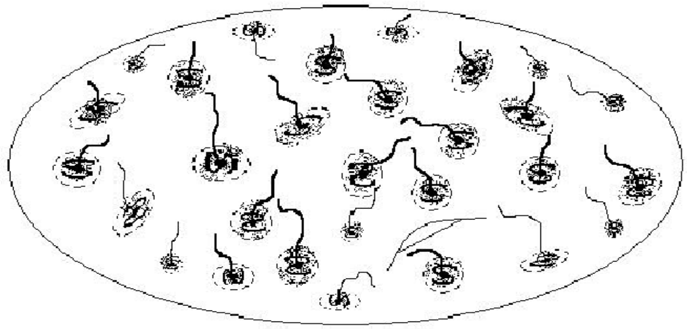

Figure 1: A schematic representation of the cosmological scale of a universe satisfying assumption (C) after sufficiently long time. The emerging stationary solutions are represented with a galactic symbol enclosed in a dashed circle representing the outer spheres. The tails coming out from the inside of the emerging stationary solutions represent the large tubes developing inside possible black holes.

3.6 The averaging problem in cosmology.

We will discuss here the implications of assumption (C) for the averaging problem in cosmology. Noting that $\mathcal{N} = \frac{1+\phi}{1+\bar{\phi}}$ we rewrite the second FL equation in the form

$$\frac{a''}{a} = \frac{-\int_{\Sigma}(4\pi G(\rho+3p)+|\hat{K}|^2)(1+\phi)dv_g}{3(1+\bar{\phi})V} = \mathcal{H}^2\frac{\bar{\phi}}{1+\bar{\phi}}. \tag{83}$$

with

$$\bar{\phi} = \frac{-\int_{\Sigma}(4\pi G(\rho+3p)+|\hat{K}|^2)(1+\phi)dv_g}{3\mathcal{H}^2 V} \tag{84}$$

The integrand is the same as in equation 80 therefore we can decompose the integration as we did in equation 81. Note that if in equation 80 we write $3\mathcal{H}^2\int_{\Sigma}\phi dv_g = -3t\mathcal{V}\bar{\phi}$, we get because of item four in assumption (C) and the fact that the reduced volume is monotonically decreasing and bounded below by V_H that $\bar{\phi} = \frac{E_{CMC}}{-3t\mathcal{V}} + O(t^{-(2+\epsilon)}) = O(1/t)$. This gives the estimation that the factor $1+\bar{\phi}$ in the denumerator of equation 83 behaves as $1+O(t^{-1})$. All together gives

$$\frac{1}{a}\frac{d^2a}{d\tau^2} = -\frac{4\pi G(\bar{\mathcal{M}}_{ADM}+\bar{\mathcal{R}})}{3-4\pi G\mathcal{H}^{-2}(\bar{\mathcal{M}}_{ADM}+\bar{\mathcal{R}})} + O(t^{-(4+\epsilon)}) = -\frac{4\pi G(\bar{\mathcal{M}}_{ADM}+\bar{\mathcal{R}})}{3} + O(t^{-(3+\epsilon)}) \tag{85}$$

where $\bar{\mathcal{M}}_{ADM}$ is the volume average of the sum of the ADM masses of the emerging stationary solutions, and $\bar{\mathcal{R}} = \bar{\rho}_{rad} + 3\bar{p}_{rad} + \bar{\rho}_G + 3\bar{p}_G$ where $\bar{\rho}_{rad}$, $\bar{p}_{rad}$ and $\bar{\rho}_G$, $\bar{p}_G$ are the volume average of the energy and pressure densities of material and gravitational radiation respectively. Equation 85 is a differential equation in τ, however the estimate on its right hand side is in terms of $t = 1/\mathcal{H}$. We

thus complement this equation with a differential equation for τ as a function of t. From the defining equation of τ we get the equation

$$\frac{d\tau}{dt} = 1 + \bar{\phi} = 1 - \frac{4\pi G}{3} t^2 (\bar{\mathcal{M}}_{ADM} + \bar{\mathcal{R}}). \tag{86}$$

Equations 85 and 86 are the main equations for the averaging problem under assumption (C) and asymptotically in time. We remark that still the Einstein equations have to be used in full, to provide an estimation of the radiative term $\bar{\mathcal{R}}$. The next section intends to provide these estimates in the case $\mathcal{M}_{ADM} = 0$, i.e. a purely radiative solution.

4 Long time smoothing and estimates on the gravitational energy: radiation.

We will use the notation H^s for the Sobolev space with s derivatives and $H^s_{g_H}$ for the Sobolev space where the norms and covariant derivatives are calculated via g_H (see [3]). We will prove here Theorem 1. The proof is a natural extension of the analysis in [6].

Theorem 2 (Expansive smoothing and energy estimates). *Let Σ be a compact and rigid hyperbolic manifold. There is an $\epsilon > 0$ such that the Einstein CMC flow of a cosmologically scaled initial state (i.e. with $\mathcal{H} = 1$) (g, K) with $\mathcal{V} - \mathcal{V}_{inf} \leq \epsilon$ and $\tilde{\mathcal{E}}_1 \leq \epsilon$ has the following long time properties (take $t = \frac{1}{\mathcal{H}}$):*

1. *The limit $lim_{t\to\infty} t^3 Q_0$ is finite and greater than zero.*
2. *there are $n_i \geq 0$ such that $\lim_{t\to\infty} \frac{t^{2i+3}}{(\ln t)^{n_i}} Q_i \leq \infty$ for $i \geq 1$.*
3. *for given $\gamma > 0$, $\int_t^\infty \frac{\int_\Sigma |\hat{K}|^2 dv_g}{u} du \geq C t^{-(2+\gamma)}$.*
4. *$|\hat{K}|^2 \leq C t^{-4}$ pointwise (not volume averaged).*

In particular the cosmologically scaled flow of a $H^i \times H^{i-1}$ state (for any $i \geq 1$) as in the hypothesis above converges in $H^i \times H^{i-1}$ to the canonical flat cone state $(g, K) = (g_H, -g_H)$.

Proof of theorem 1. We start by recalling a result from [6] that will be useful to prove items 1 and 2 in theorem 1.

Lemma 1 1 *Let Σ be a compact and rigid hyperbolic manifold. There are C and ϵ_0 such that if a cosmologically normalized CMC state (g, K), where g is harmonic with respect to g_H, is ϵ-close to $(g_H, -g_H)$ in the $H^3_{g_H} \times H^2_{g_H}$ topology, with $\epsilon \leq \epsilon_0$ then there is a constant C (dependent on ϵ_0) such that*

$$C^{-1} \tilde{\mathcal{E}}_1 \leq (\|g - g_H\|^2_{H^3_{g_H}} + \|K + g_H\|^2_{H^2_{g_H}}) \leq C \tilde{\mathcal{E}}_1. \tag{87}$$

We get therefore the elliptic estimate for the Newtonian potential $\phi = \hat{N} = \frac{k^2 N}{3} - 1$ from the lapse equation

$$\|\hat{N}\|_{H^2_{g_H}} \leq C\|\hat{K}\|_{H^2_{g_H}}\|\hat{K}\|_{L^2_{g_H}} \leq C\tilde{\mathcal{E}}_1, \tag{88}$$

and

$$\|\hat{N}\|_{H^3_{g_H}} \leq C\|\hat{K}\|_{H^2_{g_H}}\|\hat{K}\|_{H^1_{g_H}} \leq C\tilde{\mathcal{E}}_1. \tag{89}$$

To extract conclusions on the decay of the Sobolev norms of the cosmologically normalized states we will make use of the fact proved in [3] that under the conditions of the last lemma, ϵ_0 and $\tilde{\mathcal{E}}_{i-1}$ controls the difference of the states in $H^i_{g_H} \times H^{i-1}_{g_H}$ with respect to the background state $(g_H, -g_H)$ states at zero, i.e. the derivatives tend to zero in $L^2_{g_H}$ as ϵ_0 and $\tilde{\mathcal{E}}_i$ tend to zero.

Item 1. The Gauss equation gives the following inequality for the evolution of the first order cosmologically normalized Bel-Robinson energy ([6])

$$\frac{d\tilde{\mathcal{E}}_1}{d\sigma} \leq -2\tilde{\mathcal{E}}_1 + C\tilde{\mathcal{E}}_1^{\frac{3}{2}}. \tag{90}$$

with c a constant greater than zero. It follows therefore that $\tilde{\mathcal{E}}_1$ decays faster than the solution $x(\sigma)$ to the following ordinary differential equation and same initial condition

$$x' = -2x + cx^{\frac{3}{2}}. \tag{91}$$

This is a Bernoulli type of equation that can be solved by making the change of variables $v = x^{-\frac{1}{2}}$ which gives the differential equation

$$v' = v - \frac{c}{2}, \tag{92}$$

having the solution $v = \frac{1}{2} + Ae^{\sigma}$. This implies that

$$x = \frac{x(\sigma_0)e^{-2(\sigma-\sigma_0)}}{(\frac{c}{2}(e^{-(\sigma-\sigma_0)} - 1)x(\sigma_0)^{\frac{1}{2}} + 1)^2}, \tag{93}$$

which results in the following decay of $\tilde{\mathcal{E}}_1$

$$\tilde{\mathcal{E}}_1 \leq \frac{\tilde{\mathcal{E}}_1(\sigma_0)e^{-2(\sigma-\sigma_0)}}{(\frac{c}{2}(e^{-(\sigma-\sigma_0)} - 1)\tilde{\mathcal{E}}_1(\sigma_0)^{\frac{1}{2}} + 1)^2}. \tag{94}$$

Observe that if σ_0 is big enough then we get the bound

$$\tilde{\mathcal{E}}_1 \leq \frac{\tilde{\mathcal{E}}_1(\sigma_0)e^{-2(\sigma-\sigma_0)}}{4}. \tag{95}$$

Now we prove item 1 in theorem 1. From the Gauss equation and Lemma 1 and the above estimate for $\tilde{\mathcal{E}}_1$ we get an evolution equation for $\tilde{Q}_0$ of the form

$$\frac{d\tilde{Q}_0}{d\sigma} = -2\tilde{Q}_0 + h(\sigma), \tag{96}$$

where $h(\sigma)$ is a function which is bounded in absolute value by

$$|h(\sigma)| \leq C\tilde{\mathcal{E}}_1^{\frac{3}{2}}(\sigma_0)e^{-3(\sigma-\sigma_0)}. \tag{97}$$

Therefore we get the following expression for $\tilde{Q}_0$

$$\tilde{Q}_0 = e^{-2(\sigma-\sigma_0)}(\tilde{Q}_0(\sigma_0) + e^{-2\sigma_0}\int_{\sigma_0}^{\sigma} h(u)e^{2u}du), \tag{98}$$

Clearly the integral in h has a limit when $\sigma \to \infty$. If the term in parenthesis on the right hand side has limit different than zero then we are done as then

$$\lim_{\sigma\to\infty} \frac{\tilde{Q}_0}{e^{-2\sigma}} > 0. \tag{99}$$

Let's see that the limit cannot be zero. If that happens then we have for all σ

$$\tilde{Q}_0(\sigma) = -e^{-2\sigma}\int_{\sigma}^{\infty} h(u)e^{2u}du. \tag{100}$$

The integral is negative for all σ ($\tilde{Q}_0$ is positive) and goes to zero as $\sigma \to \infty$. Then there is a diverging sequence $\{\sigma_i\}$ such that for all $\sigma \geq \sigma_i$ we have

$$-\int_{\sigma}^{\infty} h(u)e^{2u}du \leq -\int_{\sigma_i}^{\infty} h(u)e^{2u}du \tag{101}$$

making then

$$\tilde{Q}_0(\sigma) \leq \tilde{Q}_0(\sigma_i)e^{-2(\sigma-\sigma_i)}, \tag{102}$$

for all $\sigma \geq \sigma_i$. Using again the Gauss equation, Lemma 1 and the estimate above we get an evolution equation for $\tilde{Q}_0(\sigma)$ of the same form as in equation 96 with h instead bounded in absolute value by $C\tilde{\mathcal{E}}_1(\sigma_i)^{\frac{1}{2}}\tilde{Q}_0(\sigma_i)e^{-3(\sigma-\sigma_i)}$. It thus gives an expression for $\tilde{Q}_0$ of the form

$$\tilde{Q}_0(\sigma) = \tilde{Q}_0(\sigma_i)e^{-2(\sigma-\sigma_i)}(1 + e^{-2\sigma_i}\int_{\sigma_i}^{\sigma} \frac{h(u)e^{2u}}{\tilde{Q}_0(\sigma_i)}du). \tag{103}$$

To see that $\lim_{\sigma\to\infty}\tilde{Q}_0e^{2\sigma} > 0$ we note the following bound for the integral term in the equation 103 above

$$|e^{-2\sigma_i}\int_{\sigma_i}^{\infty} \frac{h(u)e^{2u}}{\tilde{Q}_0(\sigma_i)}du| \leq C\tilde{\mathcal{E}}_1(\sigma_i)^{\frac{1}{2}}, \tag{104}$$

which tends to zero as $\sigma_i \to \infty$. This is a contradiction, thus the limit must be positive.

Item 2. Now we prove item 2. By induction we will be able to get an equation for $\tilde{\mathcal{E}}_i(\sigma)$ of the form

$$\tilde{Q}_i' = -(2 + h'(\sigma))\tilde{Q}_i + h(\sigma)\tilde{Q}_i^{\frac{1}{2}}, \tag{105}$$

where $h'(\sigma)$ and $h(\sigma)$ are functions bounded in absolute value by $C'\sigma^{n'}e^{-\sigma}$ and $C\sigma^n e^{-\sigma}$ for some C', C and n', n constants. It follows after making the change of variable $v = \tilde{Q}_i^{\frac{1}{2}}$ that $\tilde{Q}_i$ can be bounded by an expression of the form

$$\tilde{Q}_i \leq C\sigma^{2(n+1)}e^{-2\sigma}, \tag{106}$$

for some constant C.

Lemma 2 *Suppose that a solution to the CMC flow (g, K) has*

$$\tilde{Q}_j(\sigma) \leq C_j \sigma^{n_j} e^{-2\sigma}, \tag{107}$$

for $j = 0, \ldots, i \geq 1$, then $\tilde{Q}_{i+1}$ satisfies an equation of the form 105 and therefore satisfies an asymptotic of the form 107 for $j = i+1$.

Proof: We start with the differential inequality for $\tilde{Q}_i$. Make $\beta = \frac{-3}{k}$. Then $Q_i(k) = \lambda^{(2i-1)}Q_i(k)$, and therefore

$$\frac{d\tilde{Q}_i}{d\sigma} = \frac{3}{\beta}\frac{d\tilde{Q}_i}{dk} = \frac{3}{\beta}((2i+1)\frac{\beta^{2i+2}}{3}Q_i + \beta^{2i+1}\frac{dQ_i}{dk}). \tag{108}$$

A useful trick for the calculations that follow is to write

$$\beta^{2i+1}\frac{dQ_i}{dk} = \beta\frac{dQ_i(\beta^{-2}\mathbf{g})}{d(\beta k)} \tag{109}$$

where the β inside the derivative on the right hand side is taken constant equal to its value at the time of differentiation. Thus we are calculating the k-derivative of the cosmologically scaled solution at $k = -3$. Putting all together we get

$$\frac{d\tilde{Q}_i}{d\sigma} = (2i+1)\tilde{Q}_i + 3\frac{dQ_i(\beta^{-2}\mathbf{g})}{d(\beta k)}. \tag{110}$$

We are going to study the derivatives $\frac{dQ_i}{dk}$ of perturbation of the canonical flat cone state $(g_H, -g_H)$ at $k = -3$. From the Gauss equation we have

$$\frac{dQ_{(i)}}{dk} = -3\int_\sigma NQ_{(i)abTT}\Pi^{ab}dv_g - \int_\Sigma 2N(E^{ab}_{(i)}J_{(i)aTb} + B^{ab}_{(i)}J^*_{(i)aTb})dv_g, \tag{111}$$

therefore

$$3\beta^{2i+1}\frac{dQ_i}{dk} = -9\int_\sigma \tilde{N}\tilde{Q}_{(i)ab\tilde{T}\tilde{T}}\tilde{\Pi}^{ab}dv_{\tilde{g}} - \int_\Sigma 6\tilde{N}(\tilde{E}^{ab}_i\tilde{J}_{(i)a\tilde{T}b} + \tilde{B}^{ab}_i\tilde{J}^*_{(i)a\tilde{T}b})dv_{\tilde{g}}. \tag{112}$$

We will say that a term is an $\mathcal{O}(\sigma)$ if it can be bounded in absolute value by a term of the form $C\sigma^n e^{-\sigma}$ for some natural number n. Let's start by analyzing the first term on the right hand side of equation 111. Making

$$\hat{\Pi}_{ab} = \Pi_{ab} + \frac{k}{3}(\mathbf{g}_{ab} + T_aT_b),\ \hat{N} = N - \frac{3}{k^2}, \tag{113}$$

we get

$$-9\int_{\Sigma}\tilde{N}Q_{abTT}\tilde{\hat{\Pi}}^{ab}dv_{\tilde{g}}-3\tilde{Q}_i-9\int_{\Sigma}\tilde{\hat{N}}\tilde{Q}_i dv_{\tilde{g}}. \tag{114}$$

Using Lemma 1 and the estimate on $\tilde{\mathcal{E}}_1$ above we get the term

$$-3\tilde{Q}_i+\mathcal{O}(\sigma)\tilde{Q}_i. \tag{115}$$

Now we estimate the second term in equation 111, and therefore we need estimates of $\tilde{J}$ and $\tilde{J}^*$. We will do the calculations only for J, those for J^* proceed in exactly the same way. We note first the following inductive formula for J

$$J(\mathbf{W}_i)_{abc}=\hat{\Pi}^{de}\boldsymbol{\nabla}_e\mathbf{W}_{(i-1)dabc}-\tfrac{k}{3}\mathbf{W}_{(i)dabc}T^d+T*\mathbf{Rm}*\mathbf{W}_{i-1}+\boldsymbol{\nabla}_T J(\mathbf{W}_i)_{abc} \tag{116}$$

where the $*$ is some tensorial multiplication whose particular form is not important to our purposes. We can write the formula above symbolically as

$$J(\mathbf{W}_i)=\hat{\Pi}*\boldsymbol{\nabla}\mathbf{W}_{i-1}-\tfrac{k}{3}\mathbf{W}_i*T-\tfrac{k}{3}J(\mathbf{W}_{i-1})+T*\mathbf{Rm}*\mathbf{W}_{i-1}+\boldsymbol{\nabla}_T J(\mathbf{W}_{i-1}). \tag{117}$$

Now, inducting the fifth term on the first, second, third and fourth gives the following terms respectively

1. $$\sum_{j=0}^{j=i-1}\boldsymbol{\nabla}_T^j(\hat{\Pi}*\boldsymbol{\nabla}\mathbf{W}_{i-1-j}) \tag{118}$$

2. $$\sum_{j=0}^{j=i-1}\boldsymbol{\nabla}_T^j(\tfrac{-k}{3}*T*\mathbf{W}_{i-j}), \tag{119}$$

3. $$\sum_{j=0}^{i-2}\boldsymbol{\nabla}_T^j(\tfrac{-k}{3}J(\mathbf{W}_{i-(j+1)})), \tag{120}$$

4. $$\sum_{j=0}^{i-1}\boldsymbol{\nabla}_T^j(T*\mathbf{Rm}*\mathbf{W}_{i-1-j}). \tag{121}$$

The only terms that are not going to count as $\mathcal{O}(\sigma)$ or $\mathcal{O}(\sigma)\tilde{Q}_i^{\frac{1}{2}}$ are those coming from the expression 2 and when the $\boldsymbol{\nabla}_T$ derivative applies only to the $\mathbf{W}_{i-j}$ giving

$$\frac{-k}{3}i\mathbf{W}_i*T \tag{122}$$

When we take into accound this and a similar term aroused from a formula for J^* and plug them into equation 111 we get a contribution of the form

$$-2i\tilde{Q}_i \tag{123}$$

As said above and as we will explain in a moment all other terms are going to count as $\mathcal{O}(\sigma)$ or $\mathcal{O}(\sigma)\tilde{Q}_i^{\frac{1}{2}}$ therefore we would get, putting equations 110,115 and the last estimate together we get

$$\frac{d\tilde{Q}_i}{d\sigma} = -(2+\mathcal{O}(\sigma))\tilde{Q}_i + \mathcal{O}(\sigma)\tilde{Q}_i^{\frac{1}{2}}, \tag{124}$$

as we wanted in the induction. To discuss the other terms then we start by recalling some propositions from [3] restated in a different form for convenience of the article.

Lemma 3 *Let (g,K) be a CMC flow on a rigid hyperbolic manifold Σ. Suppose that the initial cosmological state is ϵ-close to the standard flat cone state $(g_H, -g_H)$ as in Lemma 1 then (all derivatives below are taken at $k=-3$)*

1. $$\|\nabla_T^i \Pi\|_{H^j_{g_H}},\ i\geq 1,\ j=0,1,2, \tag{125}$$
   are controlled by $\mathcal{E}_{i+j-1}$.

2. $$(\|\nabla \mathbf{W}_i)\|_{L^2_{g_H}} + \|\mathbf{W}_i\|_{L^2_{g_H}}) \leq C(\|\mathbf{W}_{i+1}\|_{L^2_{g_H}} + \|\mathbf{W}_i\|_{L^2_{g_H}} + \|J(\mathbf{W}_i)\|_{L^2_{g_H}}) \tag{126}$$
   $i\geq 0$.

Lemma 4 $\nabla_T^h J(\mathbf{W}_i)$ *has an expression of the form*

$$\nabla_T^h J(\mathbf{W}_i) = \sum(\nabla_T^{m_1}\Pi)^{n_1} * \cdots * (\nabla_T^{m_s}\Pi)^{n_s} * \Pi^l * \nabla \mathbf{W}_k + \tag{127}$$

$$+\sum(\nabla_T^{\tilde{m}_1}\Pi)^{\tilde{n}_1} * \cdots * (\nabla_T^{\tilde{m}_s}\Pi)^{\tilde{n}_s} * \Pi^{\tilde{l}} * \nabla_T^q (T * \mathbf{Rm} * \mathbf{W}_{\tilde{k}}) \tag{128}$$

where the first sum is among the set $k\leq i+h-1$, $m_1\geq \ldots \geq m_s \geq 1$ and $\sum_j n_j(1+m_j)+l+k = i+h$, while the second is among the set $\tilde{m}_1 \geq \ldots \geq \tilde{m}_s \geq 1$ and $\sum_j \tilde{n}_j(1+\tilde{m}_j)+\tilde{k}+\tilde{l}+q = i+h-1$.

Now we prove the following Lemma.

Lemma 5 *Let (g,K) be a CMC solution. Suppose for a given value of i there are n_i and C_i such that $\tilde{\mathcal{E}}_i \leq C_i\sigma^{n_i}e^{-2\sigma} = \mathcal{O}(\sigma)$ then*

1. *there are n_i' and C_i' such that $\|\tilde{J}(\mathbf{W}_i)\|^2_{L^2_{g_H}} \leq C_i'\sigma^{n_i'}e^{-2\sigma} = \mathcal{O}(\sigma)$.*
2. *There are n_{ij}' and C_{ij}' such that $\|(\nabla_T^j J(\mathbf{W}_{i-j}))^{\sim}\|^2_{L^2_{g_H}} \leq C_{ij}'\sigma^{n_{ij}'}e^{-2\sigma} = \mathcal{O}(\sigma)$ for $j\leq i$.*

Proof: 1. Proceed by induction in i. Observe that all the factors involving Π and its time derivatives in formula 127 (with $h=0$) are controlled by $\tilde{\mathcal{E}}_i$ in $H^2_{g_H}$ by Lemma 3. The norms $\|(\nabla \mathbf{W}_k)^{\sim}\|_{L^2_{g_H}}$ are controlled using inequality 138. The second kind of terms in equation 128 are controlled as follows. The factors involving Π and its time derivatives are controlled again in $H^2_{g_H}$ by $\tilde{\mathcal{E}}_i$. The other factors can be seen as

$$(\nabla_T^q (T * \mathbf{Rm} * \mathbf{W}_{\tilde{k}}))^{\sim} = \textstyle\sum_{q_1+q_2+q_3=q} (\nabla_T^{q_1} T)^{\sim} * (\nabla_T^{q_2}\mathbf{Rm})^{\sim} * (\nabla_T^{q_3}\mathbf{W}_{\tilde{k}})^{\sim}, \tag{129}$$

$$= \sum_{q_1+q_2+q_3=q} (\boldsymbol{\nabla}_T^{q_1} T)^{\sim} * (\tilde{\mathbf{W}}_{q_2}) * (\tilde{\mathbf{W}}_{q_3+\tilde{k}}), \tag{130}$$

with $q \leq i-1$. Now Sobolev embeddings give

$$\|\tilde{\mathbf{W}}_{q_2} * \tilde{\mathbf{W}}_{\tilde{k}+q_3}\|_{L^2_{g_H}} \leq C(\|\tilde{\mathbf{W}}_{q_3}\|_{H^1_{g_H}} \|\tilde{\mathbf{W}}_{\tilde{k}+q_3}\|_{H^1_{g_H}}), \tag{131}$$

where the factors on the right are controlled by Lemma 3. The factors $(\boldsymbol{\nabla}_T T)^{\sim}$ are controlled in $H^2_{g_H}$ by Lemma 3. Finally the proof of part 2. is the same as above after using formulas 127, 128. □

The terms in 2, 3, 4 on the induction formula for J other than the ones already considered in equation 123 are easily seen to be bounded by $\mathcal{O}(\sigma)$ or $\mathcal{O}(\sigma)\tilde{Q}_i^{\frac{1}{2}}$ by the same kind of arguments as in Lemma 5. To bound the terms in 1. in the same way we need the following form of $\boldsymbol{\nabla}_T^j \boldsymbol{\nabla} \mathbf{W}_k$

$$\boldsymbol{\nabla}_T^j \boldsymbol{\nabla} \mathbf{W}_i = \sum (\boldsymbol{\nabla}_T^{m_1} \Pi)^{n_1} * \cdots * (\boldsymbol{\nabla}_T^{m_s} \Pi)^{n_s} * \Pi^l * \boldsymbol{\nabla} \mathbf{W}_k + \tag{132}$$

$$+ \sum (\boldsymbol{\nabla}_T^{\tilde{m}_1} \Pi)^{\tilde{n}_1} * \cdots * (\boldsymbol{\nabla}_T^{\tilde{m}_s} \Pi)^{\tilde{n}_s} * \Pi^{\tilde{l}} * \boldsymbol{\nabla}_T^q (T * \mathbf{Rm} * \mathbf{W}_{\tilde{k}}), \tag{133}$$

where the first sum is among the set $m_1 \geq \ldots \geq m_s \geq 1$ and $\sum_j n_j(1+m_j) + l + k = i + j$, while the second is among the set $\tilde{m}_1 \geq \ldots \geq \tilde{m}_s \geq 1$ and $\sum_j \tilde{n}_j(1+\tilde{m}_j) + \tilde{k} + \tilde{l} + q = i + j - 1$, which can be easily proved by induction by using equation

$$\boldsymbol{\nabla}_T \boldsymbol{\nabla} \mathbf{W}_i = \boldsymbol{\nabla} \mathbf{W}_{i+1} + \Pi * \boldsymbol{\nabla} \mathbf{W}_i + T * \mathbf{Rm} * \mathbf{W}_i. \tag{134}$$

This finishes the induction in Lemma 2. □.

Items 3 *and* 4. The estimate from above in item 4 comes from Lemma 1. The item 3 or the estimate from below is more involved, the argument is as follows.

Lemma 6 *For any $\epsilon > 0$ there is a ball $B_{(g_H,-g_H)}(\delta)$ of cosmologically scaled states in $H^3 \times H^2$ such that*

$$\|\tilde{N} - \frac{1}{3}\|_{L^\infty} \leq \epsilon, \tag{135}$$

and

$$4\pi G \mathcal{H} E_{CMC} \geq \frac{1}{4+\epsilon} \int_\Sigma |\hat{\tilde{K}}|^2 dv_{\tilde{g}}. \tag{136}$$

We can prove item 3 by making use of the Lemma 6. First, the derivative of the reduced volume $\mathcal{V} = \mathcal{H}^3 V$ in logarithmic time is

$$\frac{d\mathcal{V}}{d\sigma} = -3 \int_\Sigma \tilde{N} |\hat{\tilde{K}}|^2 dv_{\tilde{g}}. \tag{137}$$

If we integrate it from σ to ∞ and use lemma 6 above we get the following inequality

$$\frac{1}{4+\epsilon} \int_\Sigma |\hat{\tilde{K}}|^2 dv_{\tilde{g}} \leq 4\pi G \mathcal{H} E_{CMC} = 3 \int_\sigma^\infty (\int_\Sigma \tilde{N} |\hat{\tilde{K}}|^2 dv_{\tilde{g}}) d\sigma \leq (1+\epsilon) \int_\sigma^\infty (\int_\Sigma |\hat{\tilde{K}}|^2 dv_{\tilde{g}}) d\sigma. \tag{138}$$

Making $U = \int_\sigma^\infty (\int_\Sigma |\hat{\tilde{K}}|^2 dv_{\tilde{g}}) d\sigma$ the inequality 138 is written as

$$U' \geq -(4+\epsilon)(1+\epsilon)U, \tag{139}$$

which after integration gives the left hand side inequality in item 3.

Proof of Lemma 6. First we note that the estimate for $\tilde{N} - \frac{1}{3}$ is deduced from Lemma 1. For the second estimate it may be deduced from the calculation of the Hessian of the energy that we did before, however we will follow a direct estimate from the Lichnerowicz equation. We argue as follows. Say $g = \phi^4 g_Y$ where g_Y is the unique metric in the conformal class of g having scalar curvature -6. Then ϕ satisfies

$$-\Delta\phi + \frac{3}{4}(\phi^5 - \phi) = \frac{1}{8}\phi^{-3}|\hat{\tilde{K}}|^2_Y. \tag{140}$$

The maximum principle gives $\phi \geq 1$. Making $\bar{\phi} = \phi - 1$ rewrite equation 140 as

$$-\Delta\bar{\phi} + \frac{3}{4}\phi(\phi^3 + \phi^2 + \phi + 1)\bar{\phi} = \frac{|\hat{\tilde{K}}|^2_Y}{8\phi^3}. \tag{141}$$

At the point where ϕ or $\bar{\phi}$ is maximum we have

$$\bar{\phi} \leq \frac{1}{12}\frac{|\hat{\tilde{K}}|^2\phi^4}{\phi^3 + \phi^2 + \phi + 1} \leq \frac{|\hat{\tilde{K}}|^2\phi}{12}, \tag{142}$$

which gives if $\|\hat{K}\|_{L^\infty_g}$ is small

$$\|\bar{\phi}\|_{L^\infty} \leq \frac{\|\hat{K}\|^2_{L^\infty_g}}{12 - \|\hat{K}\|^2_{L^\infty_g}}. \tag{143}$$

Also note that

$$-\sigma(\Sigma) \leq -6V_Y^{\frac{2}{3}}, \tag{144}$$

which gives

$$0 \leq \int_\Sigma (\phi^6 - 1) dv_Y \leq V - V_H. \tag{145}$$

Writing $\phi^6 - 1 = (\phi - 1)(\phi^5 + \phi^4 + \phi^3 + \phi^2 + \phi + 1)$ we get

$$6\int_\Sigma (\phi - 1) dv_{g_Y} \leq V - V_H. \tag{146}$$

Integrating equation 140 we get

$$6\int_\Sigma (\phi^5 - \phi) dv_{g_Y} = \int_\Sigma \phi^{-3}|\hat{\tilde{K}}|^2_Y dv_{g_Y}. \tag{147}$$

Under the assumptions we have and using equation 143 we can get from equation 147 above the inequality

$$6(4+\epsilon)\int_{\Sigma}(\phi-1)dv_{g_Y} \geq \int_{\Sigma}\phi^{-2}|\hat{\hat{K}}|^2_Y dv_{g_Y} = \int_{\Sigma}|\hat{\hat{K}}|^2 dv_g. \tag{148}$$

which together with equation 81 gives the inequality

$$(4+\epsilon)(V-V_H) \geq \int_{\Sigma}|\hat{\hat{K}}|^2 dv_g. \tag{149}$$

as desired. □.

This finishes theorem 1. □.

5 States of arbitrarily large gravitational energy.

We will construct a one parameter family of states (g_λ, K_λ) such that

1. $k_\lambda = k_0$ fixed,
2. $Vol_{g_\lambda} \rightarrow_{\lambda\rightarrow\infty} \infty$ and $\|\hat{K}_\lambda\|_{L^2_{g_\lambda}} \rightarrow_{\lambda\rightarrow\infty} \infty$,
3. The "big-bang" family of states, i.e. the volume-one normalized family of states above has

$$-k_\lambda \rightarrow \infty, \tag{150}$$

$$Vol_{g_\lambda}(\Sigma) = 1, \tag{151}$$

$$\lim_{\lambda\rightarrow\infty}\|\hat{K}_\lambda\|_{L^2_{g_\lambda}} = \infty. \tag{152}$$

As has been argued above, these states represents a one parameter family of states with arbitrarily large gravitational energy. The construction is as follows. Pick the hyperbolic metric g_H and a non zero transverse traceless tensor $\hat{K}$ with respect to it. According to the conformal method it is possible to find a solution to the constraint of the form $(g_\lambda, K_\lambda) = (\varphi^4 g_H, \lambda^2\varphi^{-2}\hat{K} - \varphi^4 g_H)$ (the mean curvature being $k = k_0 = -3$ and one parameter family of states as above with arbitrary k_0 can be obtained by scaling), by solving the elliptic equation

$$\Delta\varphi = -\frac{3}{4}\varphi - \frac{\lambda^4}{8}|\hat{K}|^2_{g_H}\varphi^{-7} + \frac{3}{4}\varphi^5. \tag{153}$$

Now we prove items 2. Multiplying equation 153 by φ and integrating we get

$$\frac{\lambda^4}{8}\int_{\Sigma}|\hat{K}|^2_{g_H}\varphi^{-6}dv_{g_H} = \int_{\Sigma}|\nabla\varphi|^2 + \frac{3}{4}(\varphi^6 - \varphi^2)dv_{g_H}. \tag{154}$$

Note that the left hand side is $\frac{1}{8}\|\hat{K}_\lambda\|^2_{L^2_{g_\lambda}}$. If the left hand side doesn't diverge as $\lambda\to\infty$ then the right hand side remains bounded in particular the $H^1_{g_H}$ norm of φ remains bounded. Pick an open set Ω where $|\hat{K}|_{g_H}\geq \epsilon>0$. Then as φ is bounded in H^1 we have $Vol\{x\in\Omega/\varphi(x)<n\}\to Vol(\Omega)$ as $n\to\infty$ uniformly in λ. Then for some n we have $Vol\{x\in\Omega/\varphi(x)<n\}>\frac{Vol(\Omega)}{2}$ uniformly in λ, and so the left hand side is bigger than $\frac{\lambda^4}{16n^6}\epsilon^2 Vol(\Omega)$ which diverges when $\lambda\to\infty$ which is a contradiction. This proves item 2, to prove item 3 we argue as follows. The L^2 norm of $\hat{K}_\lambda$ of the volume one states are

$$\frac{\lambda^4\int_\Sigma|\hat{K}|^2\varphi^{-6}dv_{g_H}}{(\int_\Sigma\varphi^6 dv_{g_H})^{\frac{1}{3}}}=\frac{\int_\Sigma|\nabla\varphi|^2+\frac{3}{4}(\varphi^6-\varphi^2)dv_{g_H}.}{(\int_\Sigma\varphi^6 dv_{g_H})^{\frac{1}{3}}}. \tag{155}$$

We have that an upper bound on the left hand side in the last equation implies an upper bound for the H^1 norm of φ, for if not we have $\int_\Sigma\varphi^6 dv_{g_H}\to\infty$ which would make the numerator of the right hand side diverging in λ, but we know $\int_\Sigma\varphi^6 dv_{g_H}$ diverges which is a contradiction.

6 Summary and open questions.

We have introduced the notion of general $\mathcal{K}=-1$ cosmological model as a formal definition allowing to study cosmological notions in arbitrary solutions of the Einstein equations. This gave us a framework to study general cosmological solutions in a cosmological language. The approach may be applicable to models other than general $\mathcal{K}=-1$ cosmological models, i.e. models with different spatial topologies. Thinking on the *averaging problem in cosmology* we have defined volume-averaged cosmological parameters and an averaging map: a correspondence between arbitrary solutions and homogeneous and isotropic Lorentzian spaces. Those concepts allowed us to give a precise mathematical formulation of the averaging problem in cosmology. In another section and aiming at the start of a rigorous analysis of cosmological evolution from the solutions at the natural scale, (i.e. including the small scale), we have introduced *assumption (C)* which precisely describe a certain class of solutions. Those solutions are divided into two main subclasses: radiative and mass gap. We have given a detailed description of the full structure of the radiative solutions. We have also analyzed the averaging problem in cosmology in precise quantitative terms for mass gap solutions. The attemp may be considered as a first step towards the ideal goal of attacking the averaging problem in cosmology directly from the solutions at the small scale. Finally we constructed initial "big-bang" states of arbitrarily large gravitational energy, showing that, apriori there is no mathematical restriction to assume the gravitational energy to be low at the beginnings of time.

There are several questions and avenues of research left open in the present article, of varying difficulty however. For instance one may want to see in action the formalism of *general cosmological models* in cosmological solutions with Cauchy surfaces of non hyperbolic topology. Also and perhaps more important is to obtain rigorous results that may support or not *assumption (C)*. Any rigorous

result of the sort would put the study of the averaging problem in cosmology from the small scale on a firm basis. Analyzing the validity of assumption C from the Einstein equations is a very difficult problem. A central point is to study the spatial asymptotic of stationary solutions that may emerge in time. Is an emerging stationary solution necessarily spatially asymptotically flat in the long time?. If the answer is affirmative one may be in a better position to prove the a priori estimates in assumption (C). The answer may instead be negative and that would open a new avenue of research. Finally the analysis of the validity of the averaging problem in cosmology from assumption C was only asymptotic in time, and therefore of non obvious applicability. An interesting question is to study the validity of the analysis but in finite times.

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